Lecture 3

6.3* - The Natural Exponential Function

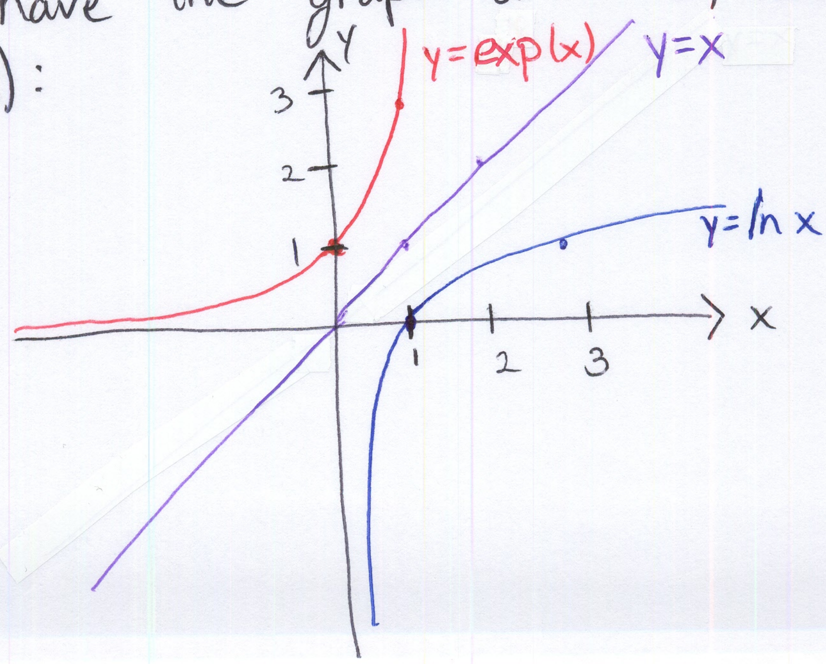
We have the function $f(x) = \ln x$ which is

- one-to-one
- differentiable
- $\mathcal{D}(\ln x) = (0, \infty)$
- $\mathcal{R}(\ln x) = (-\infty, \infty)$

This means it has an inverse function, which we will call $f^{-1}(x) = \exp(x)$. Since $f(x)$ has the above properties, $f^{-1}(x)$ is

- one-to-one
- differentiable
- $\mathcal{D}(\exp x) = (-\infty, \infty)$
- $\mathcal{R}(\exp x) = (0, \infty)$

Since we have the graph of $\ln x$, we can also graph $\exp(x)$:



To get a better idea of what exp(x) is, let's look at the chart:

$y=\ln x$	$0=\ln 1$	$1=\ln e$	$2=\ln e^2$	$r=\ln e^r$
$x=\exp(y)$	$1=\exp(0)$	$e=\exp(1)$	$e^2=\exp(2)$	$e^r=\exp(r)$

This means we can write $f(x)=\exp(x)$ with a more familiar notation:

$f(x)=e^x$ (e to the power x)

We can use that these functions are inverses to solve equations:

Ex: Solve for x if $\ln(2x+1)=3$

$$e^{\ln(2x+1)}=e^3 \Rightarrow 2x+1=e^3 \Rightarrow \boxed{x=\frac{e^3-1}{2}}$$

Ex: Solve for x if $e^{\frac{x+1}{3}}=7$.

$$\ln\left(e^{\frac{x+1}{3}}\right)=\ln 7 \Rightarrow \frac{x+1}{3}=\ln 7 \Rightarrow x+1=3\ln 7=\ln 7^3$$

$$\Rightarrow \boxed{x=\ln(7^3)-1=\ln(343)-1}$$

Analyzing the graph of $f(x)=e^x$, we see that

$$\lim_{x\to -\infty} e^x = 0 \qquad \& \lim_{x\to\infty} e^x = \infty$$

Ex: Compute $\displaystyle\lim_{x\to\infty} \frac{e^x}{2e^x-1}$

$$\lim_{x\to\infty} \frac{e^x}{2e^x-1}\cdot\frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \lim_{x\to\infty}\frac{1}{2-\frac{1}{e^x}} = \frac{1}{2-0} = \boxed{\frac{1}{2}}$$

From the algebraic properties of ln, we get some for e^x:

$$e^{x+y}=e^xe^y \qquad e^{x-y}=\frac{e^x}{e^y} \qquad (e^x)^y=e^{xy}$$

Ex: Simplify $\displaystyle\frac{e^{x^2}e^{2x}}{(e^{x+1})^2} = \frac{e^{x^2+2x}}{e^{2x+2}} = e^{x^2+2x-(2x+2)}$

$$=\boxed{e^{x^2-2}}$$

We can find the derivative of $f(x)=e^x$ using logarithmic differentiation:

$$y=e^x \Leftrightarrow \ln y = x \Rightarrow \frac{1}{y}\frac{dy}{dx}=1 \Leftrightarrow \frac{dy}{dx}=y=e^x$$

And so: $\frac{d}{dx}e^x=e^x$ & $\frac{d}{dx}e^{g(x)}=g'(x)e^{g(x)}$

Also, these give us the integrals:

$$\int e^x\,dx=e^x+C \quad \& \int g'(x)e^{g(x)}\,dx=e^{g(x)}+C$$

<u>Ex</u>: Compute $\frac{d}{dx}\sin^2\left(e^{x^2+1}\right)$

$$\frac{d}{dx}\left[\sin^2\left(e^{x^2+1}\right)\right]=2\sin\left(e^{x^2+1}\right)\cdot\frac{d}{dx}\left[\sin\left(e^{x^2+1}\right)\right]$$

$$=2\sin\left(e^{x^2+1}\right)\cdot\cos\left(e^{x^2+1}\right)\cdot\frac{d}{dx}\left[e^{x^2+1}\right]=4x\sin\left(e^{x^2+1}\right)\cos\left(e^{x^2+1}\right)e^{x^2+1}$$

<u>Ex</u>: Compute $\int \sec^2 x\, e^{\tan x}\,dx$

$$\int \sec^2 x\, e^{\tan x}\,dx \underset{du=\sec^2 x\,dx}{\overset{u=\tan x}{=\!=}} \int e^u\,du=e^u+C$$

$$=e^{\tan x}+C$$